Benha University
Faculty of Science
Dept. of Mathematics

Time : 2 Hours
Second Term 2015-2016
Date : 29/5/2016

Complex Variables (423 M) for Fourth Year Students,
Faculty of Education (Math Section)

جامعة بنها – كلية العلوم – قسم الرياضيات

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الممتحن: د . / محمد السيد أحمد حسن نصر

مدرس بقسم الرياضيات بكلية العلوم

الامتحان+ نموذج إجابة

ورقة كاملة

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Answer the following questions

Question 1.

(a) Obtain all values of $\log(1-i\sqrt{3})$ & $(1+i)^i$.

(b) Show that $\sin \bar{z}$ isn't an analytic function of z anywhere.

Question 2.

(a) Use Cauchy's integrals to evaluate:

$$I=\int_C \frac{z^2\, e^{\pi z}}{(z-i)^2}\,dz, \qquad J=\int_C \frac{z^2+e^{z^2}}{z^3}\,dz, \qquad K=\int_C \frac{z^2\, e^{3z}}{(z-i\frac{\pi}{2})^2}\,dz.$$

where C is the circle $|z-i|=2$.

(b) If $w=u+\ i\,v$ is analytic where $u=2x(1-y)$, find v then obtain w & $\dfrac{dw}{dz}$.

Question 3.

Obtain expansions in powers of $(z-1)$ for the function $f(z)=\dfrac{z}{(z-1)(z-3)}$ inside and outside circle $|z-1|=2$.

Question 4.

Use contour integration to evaluate the following integrals:

$$L=\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)}, \qquad M=\int_0^\infty \frac{\cos x\,dx}{(x^2+1)^2}, \qquad N=\int_{-\pi}^{\pi} \frac{\cos\theta\, d\theta}{5+4\cos\theta}.$$

Good Luck!

د/ محمد السيد نصر

The answer

Answer Question 1:

(a) Obtain all values of $\log(1-i\sqrt{3})$ & $(1+i)^i$.

$$\log(1-i\sqrt{3}) = \log\left[2e^{-i\frac{\pi}{3}}\right] = \ln 2 + i(2\pi k - \frac{\pi}{3}), \quad k = 0,1,2,3,\ldots.$$

(b) Show that $\sin\bar{z}$ isn't an analytic function of z anywhere.

$$\sin\bar{z} = \sin(x-iy) = \sin x\cos iy - \cos x\sin iy$$

$$\sin\bar{z} = \sin x\cosh y - i\cos x\sinh y = u + iv$$

$$\Rightarrow \quad u = \sin x\cosh y \quad \& \quad v = -\cos x\sinh y$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \cos x\cosh y \quad \& \quad \frac{\partial v}{\partial y} = -\cos x\cosh y,$$

$$\frac{\partial u}{\partial y} = \sin x\sinh y \quad \& \quad \frac{\partial v}{\partial x} = \sin x\sinh y$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial y}$$

CRC aren't satisfied

So, $\sin\bar{z}$ isn't an analytic function.

Answer Question 2:

(a) $$I=\int_C \frac{z^2 e^{\pi z}}{(z-i)^2}\,dz = 2\pi i\left[\frac{d}{dz} z^2 e^{\pi z}\right] = 2\pi i[(2z+\pi z^2)e^{\pi z}]_{z=i}$$

$$= 2\pi i[(2i-\pi)e^{\pi i}] = 2\pi i[(\pi-2i)] = 2\pi(2+i\pi),$$

$$J=\int_C \frac{z^2+e^{z^2}}{z^3}\,dz = \frac{2\pi i}{2}\left[\frac{d^2}{dz^2}(z^2+e^{z^2})\right]_{z=0} = \pi i\left[\frac{d}{dz}(2z+2z\,e^{z^2})\right]_{z=0}$$

$$= \pi i(2+2e^{z^2}+4z^2e^{z^2})_{z=0} = \pi i(2+2) = 4\pi i,$$

$$K=\int_C \frac{z^2 e^{3z}}{(z-i\frac{\pi}{2})^2}\,dz = 2\pi i\left[\frac{d}{dz} z^2 e^{3z}\right] = 2\pi i[(2z+3z^2)e^{3z}]_{z=i\frac{\pi}{2}}$$

$$= 2\pi i\left[(\pi i-\frac{3\pi^2}{4})(-i)\right] = 2\pi^2(i-\frac{3\pi}{4}) = \frac{\pi^2}{2}(-3\pi+4i)$$

(b) $\because$ w is analytic $\Rightarrow \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \Rightarrow \dfrac{\partial v}{\partial x} = 2x$

$$\Rightarrow \quad v=\int 2x\,dx+\phi(y) = x^2+\phi(y)$$

$$\because \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \Rightarrow \frac{\partial u}{\partial x}=2(1-y) \Rightarrow 2(1-y)=\frac{\partial}{\partial y}\left(x^2+\phi(y)\right)=\phi'(y)$$

$$\Rightarrow \phi(y)=2y-y^2 \quad \Rightarrow \quad v=x^2+2y-y^2$$

$$w=u+iv=2x-2xy+i(x^2+2y-y^2)$$

$$=2(x+iy)+i(x^2-y^2+2ixy)=2z+iz^2$$

$$\frac{dw}{dz}=\frac{1}{i}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=i\,2x+2-2y=2+2i(x+iy)=2+2iz.$$

Answer Question 3:

(a) at $|z-1|<2$

$$f(z)=\frac{z}{(z-1)(z-3)}=\frac{1}{(z-1)}\left(1+\frac{3}{(z-1-2)}\right)$$

$$=\frac{1}{z-1}+\frac{3}{(z-1)}\left(\frac{1}{-2}\right)\left(\frac{1}{1-\frac{(z-1)}{2}}\right)=\frac{1}{z-1}-\frac{3}{2(z-1)}\left(\frac{1}{1-\frac{(z-1)}{2}}\right)$$

$$=\frac{1}{z-1}-\frac{3}{2(z-1)}\left(1+\frac{(z-1)}{2}+\frac{(z-1)^2}{4}+\frac{(z-1)^3}{8}+\cdots\right)$$

$$=-\frac{1}{2(z-1)}-\frac{3}{4}-\frac{3}{8}(z-1)-\frac{3}{16}(z-1)^2-\frac{3}{32}(z-1)^3-\cdots$$

$$=-\frac{1}{2(z-1)}-3\sum_{n=0}\frac{(z-1)^n}{2^{n+2}}$$

(b) at $|z-1|>2$

$$\Rightarrow f(z)=\frac{1}{z-1}+\frac{3}{(z-1)}\left(\frac{1}{(z-1-2)}\right)=\frac{1}{z-1}+\frac{3}{(z-1)^2}\left(\frac{1}{1-\frac{2}{(z-1)}}\right)$$

$$=\frac{1}{(z-1)}+\frac{3}{(z-1)^2}\left(1+\frac{2}{(z-1)}+\frac{4}{(z-1)^2}+\frac{8}{(z-1)^3}+\cdots\right)$$

$$=\frac{1}{(z-1)}+\frac{3}{(z-1)^2}+\frac{6}{(z-1)^3}+\frac{12}{(z-1)^4}+\frac{24}{(z-1)^5}+\cdots$$

$$=\frac{1}{(z-1)}+3\sum_{n=0}\frac{2^n}{(z-1)^{n+2}}$$

Answer Question 4:

$$L=\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)}=\frac{1}{2}\int_{-\infty}^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)}$$

$$f(z)=\frac{z^2}{(z^2+1)(z^2+4)}=\frac{z^2}{(z-i)(z+i)(z-2i)(z+2i)} \quad \Rightarrow \quad z_0=\pm i,\ \pm 2i$$

$$b_1(i)=\left[\frac{z^2}{(z+i)(z-2i)(z+2i)}\right]_{z=i}=\frac{-1}{(2i)(3)}=\frac{i}{6}$$

$$b_1(2i)=\left[\frac{z^2}{(z+i)(z-i)(z+2i)}\right]_{z=2i}=\frac{-4}{(4i)(-3)}=\frac{-i}{3}$$

$$\int_C f(z)\,dz=2\pi i(\frac{i}{6}-\frac{i}{3})=2\pi(-1)(-\frac{1}{6})=\frac{\pi}{3}=2L+\int_{C'} f(z)\,dz \quad (as\ R\to\infty)$$

but $$\int_{C'} f(z)\,dz=\int_0^\pi \frac{R^2 e^{2i\theta} iRe^{i\theta}\ d\theta}{R^4(e^{2i\theta}+\frac{1}{R^2})(e^{2i\theta}+\frac{4}{R^2})}$$

$$=\frac{i}{R}\int_0^\pi \frac{e^{3i\theta}\ d\theta}{(e^{2i\theta}+\frac{1}{R^2})(e^{2i\theta}+\frac{4}{R^2})} \to 0 \ \ (as\ R\to\infty)$$

hence $\quad L=\dfrac{\pi}{6}$

$$M = \int_0^\infty \frac{\cos x\, dx}{(x^2+1)^2} = \frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{(x^2+1)^2}$$

$f(z) = \dfrac{e^{iz}}{(z^2+1)^2}$ has two poles of order two, $\quad z_0 = \pm i$

$$\int_C f(z)\, dz = 2\pi i\, b_1(i)$$

$$b_1(i) = \lim_{z\to i} \frac{d}{dz}\frac{e^{iz}}{[z+i]^2} = \left[\frac{i e^{iz}[z+i]^2 - 2e^{iz}[z+i]}{[z+i]^4}\right]_{z=i} = \frac{8}{16ei} = \frac{1}{2ei}$$

$$\int_C f(z)\, dz = 2\pi i\left(\frac{1}{2ei}\right) = \frac{\pi}{e} = \operatorname{Re}\int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{(x^2+1)^2} + \int_{C'} f(z)\, dz \quad (as\ R\to\infty)$$

But $\displaystyle \int_{C'} f(z)\, dz = \int_0^\pi \frac{e^{iR\cos\theta - R\sin\theta} R e^{i\theta}\, i\, d\theta}{(R^2 e^{2i\theta}+1)^2}$

$$= \frac{i}{R^3}\int_0^\pi \frac{e^{iR\cos\theta - R\sin\theta + i\theta} d\theta}{\left(e^{2i\theta} + \frac{1}{R^2}\right)^2} \quad \to 0 \ \ (as\ R\to\infty)$$

Hence $\quad M = \dfrac{\pi}{2e}$

$$N=\int_{-\pi}^{\pi}\frac{\cos\theta\, d\theta}{5+4\cos\theta}=\int_C \frac{(z^2+1)}{(2z)(\frac{2}{z})(z^2+\frac{5}{2}z+1)}\cdot\frac{dz}{iz}, \qquad C:|z|=1$$

$$=\frac{1}{4i}\int_C \frac{(z^2+1)}{z\,(z^2+\frac{5}{2}z+1)}\,dz,$$

$$=\frac{1}{4i}\int_C \frac{z^2+1}{z\,(z+\frac{1}{2})(z+2)}\,dz,$$

$$=(2\pi i)\frac{1}{4i}\left\{\left[\frac{z^2+1}{(z+\frac{1}{2})(z+2)}\right]_{z=0}+\left[\frac{z^2+1}{z\,(z+2)}\right]_{z=-\frac{1}{2}}\right\}$$

$$=(2\pi i)\frac{1}{4i}\left\{1-\frac{5}{3}\right\}=\frac{\pi}{2}\left\{\frac{-2}{3}\right\}=\frac{-\pi}{3}$$