جامعة بنـها - كلية العلوم – قسم الرياضيــات

الفرقة الرابعة(تربية عام - رياضيات)

الفصل الدراسي الثانــي

يوم الامتحان: الأحد 22 / 5/ 2016 م

المادة : ميكانيكا الكم (M424)

أستاذ المادة : د . / خليل محمد خليل محمد

مدرس بقسم الرياضيات بكلية العلوم

صورة من الامتحان+ نموذج إجابته

Faculty of Education — Fourth year (Math.) — 22 / 5 / 2016

Math. Department — (**Quantum Mechanics M424)** — Time: 1 hour

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Answer the following questions:

1.a	I. Find the adjoint operator $\hat{A}^{+}$ if $\hat{A}=\frac{d}{dx}$ defined on L_2 i.e. $\hat{A}\varphi(x)=\frac{d}{dx}\varphi(x)$ with the boundary condition $\varphi(\pm\infty)=0$. II. Show that any orthonormal set is linearly independent (LI).
1.b	State the postulates of quantum mechanics.
2.a	Prove that: $\underline{j}(x;t)=(\frac{\hbar}{\mu})\operatorname{Im}(\psi^{*}\frac{\partial\psi}{\partial x})$ where $\underline{j}(x;t)$ is the probability (particle) current density vector and ψ satisfy Schrödinger time dependent equation.
2.b	A particle of mass μ and energy E approaches a square potential barrier $U(x)=0,\ x<0$ and $U(x)=U_0,\ x\geq 0$ where $U_0>0$ from the left. Prove that the reflection coefficient R and the transmission coefficient T satisfy the relation $R+T=1$ in case $E>U_0$.

<u>**انظر امتحان الكهربية الديناميكية**</u>

Dr. Khalil Mohamed

إجابة السؤال 1.a:

I) Proof:

To obtain the adjoint operator, we take the inner product

$(\hat{A}\phi,\psi) = \int_{-\infty}^{\infty}(\hat{A}\phi(x))^*\psi(x)dx = \int_{-\infty}^{\infty}\frac{d}{dx}\phi(x)^*\psi(x)dx$ then by partial integration

let $u=\psi(x)$ and $dv = \frac{d}{dx}\phi(x)^* dx$ this leads to $du = d\psi(x)$, $v=\phi(x)^*$ then

$\int_{-\infty}^{\infty}\frac{d}{dx}\phi(x)^*\psi(x)dx = \psi(x)\phi(x)^*\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\phi(x)^*(\frac{d}{dx}\psi(x))dx$. From the boundary condition

$\phi(\pm\infty)=0$ then $\int_{-\infty}^{\infty}\frac{d}{dx}\phi(x)^*\psi(x)dx = 0 + \int_{-\infty}^{\infty}\phi(x)^*(-\frac{d}{dx}\psi(x))dx = (\phi,(-\frac{d}{dx})\psi) = (\phi,\hat{A}^+\psi)$

$\therefore \hat{A}^+ = -\frac{d}{dx}.$

II) Proof:

To prove that the orthonormal set is LI, let $\{\psi_i\}$ is a set of n vectors where none of which equal the zero vector and α_i are scalars. Then we make the linear combination

$$\sum_{i=1}^{n}\alpha_i\{\psi_i\}=0 \quad (1).$$

Let ψ_j an element of the set $\{\psi_i\}$, then

$$(\psi_j,\sum_{i=1}^{n}\alpha_i\{\psi_i\}) = \sum_{i=1}^{n}\alpha_i(\psi_j,\psi_i) = \sum_{i=1}^{n}\alpha_i\delta_{ij} = \alpha_j \qquad (2)$$

But $(\psi_i,0)=0$ (3)

From (2) and (3), we obtain $\alpha_i = 0 \quad \forall\ j=1,2,3,.....n$.

Then $\{\psi_i\}$ are LI.

إجابة السؤال 1.b:

*The postulates of quantum mechanics are:

1)-Postulate I: Every physical state of a dynamical system (a particle) is represented at a given instant of time t by normed vector $|\psi\rangle_t$ in H. It is assumed that the state vector contains all the information which one can know about the state of the system at that instant of time. ψ and $e^{i\delta}\psi$ where $\delta^*=\delta$ represent the same physical state.

2)- Postulate II: To every dynamical variable A there corresponds an observable $\hat{A}$. The observable $\hat{x}$ and $\hat{p}$ must satisfy $[\hat{x},\hat{p}]=i\hbar$. The rules for constructing the observable $\hat{A}$ corresponding to the dynamical variable A, in the $x-rep$ are as follows:

$(i) x \to \hat{x} = x,\ t \to \hat{t} = t,\ p \to \hat{p} = -i\hbar \frac{d}{dx}$

$(ii) A(x,p,t) \to \hat{A} = A(x, -i\hbar \frac{d}{dx}, t).$

3)- Postulate III: If a particle is in state $|\psi\rangle_t$, a measurement of a dynamical variable A which is represented by the observable $\hat{A}$ and

$\hat{A}|\varphi_n\rangle = a_n|\varphi_n\rangle,\ \langle\varphi_n|\varphi_n\rangle = \delta_{nm},\ \ \hat{1}_a = \sum_i |\varphi_i\rangle\langle\varphi_i|$ will

*yield one of the eigenvalues a_i with probability

$$\rho_\psi(a_i) = \frac{|\langle\varphi_i|\psi\rangle|^2}{\langle\psi|\psi\rangle}$$

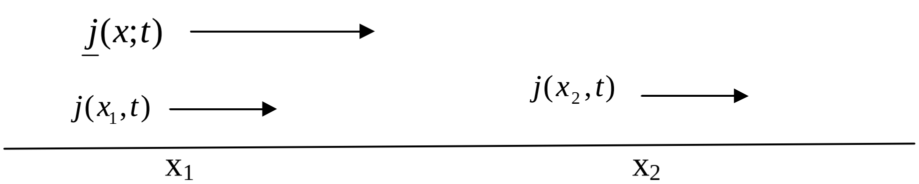
** If the result of measurement is a_k, then the state of the system will change from $|\psi\rangle$ to $|\varphi_k\rangle$ as a result of measurement.

4)- Postulate IV: The state function $\psi(x,t)$ describing the state of a dynamical system obeys the following" Schrodinger time-dependent" equation whose Hamiltonian $\hat{H}$ is

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = \hat{H}\psi(x,t)$$

إجابة السؤال 2.a:

Probability (particle) current density vector $\underline{j}(x;t)$



The probability that a particle is inside the interval (x_1, x_2) at time t is:

$$\int_{x_1}^{x_2} \rho(x,t)dx = \int_{x_1}^{x_2} \psi^*(x,t)\psi(x,t)\,dx$$

The rate of change of probability for the particle to be inside (x_1, x_2)

$$j(x_1,t) - j(x_2,t) = \frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx = \frac{d}{dt}\int_{x_1}^{x_2} \psi^*(x,t)\psi(x,t)\,dx \qquad (1)$$

$$= \int_{x_1}^{x_2} \frac{\partial}{\partial t}(\psi^*(x,t)\psi(x,t))\,dx = \int_{x_1}^{x_2} (\psi^* \frac{\partial\psi}{\partial t} + \frac{\partial\psi^*}{\partial t}\psi)\,dx$$

Since $\hat{H}\psi = i\hbar\frac{\partial\psi}{\partial t}$ and $(\hat{H}\psi)^* = -i\hbar\frac{\partial\psi^*}{\partial t}$, then

$$\frac{\partial\psi}{\partial t} = \frac{i}{\hbar}[\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} - U(x)]\psi \text{ and } \frac{\partial\psi^*}{\partial t} = -\frac{i}{\hbar}[\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} - U(x)]\psi^*; U^* = U$$

From which $\psi^*\frac{\partial\psi}{\partial t} + \frac{\partial\psi^*}{\partial t}\psi = \frac{i\hbar}{2\mu}(\psi^*\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi^*}{\partial x^2}\psi)$. Substituting into (1)

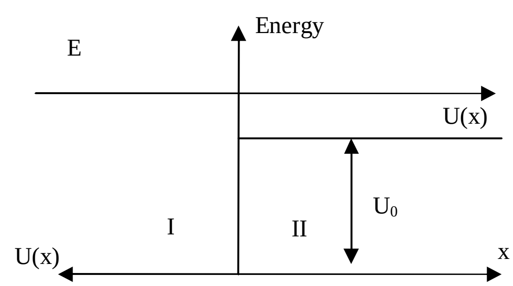
$$j(x_1,t) - j(x_2,t) = \frac{i\hbar}{2\mu}\int_{x_1}^{x_2}(\psi^*\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi^*}{\partial x^2}\psi)\,dx = \frac{i\hbar}{2\mu}\int_{x_1}^{x_2}\frac{\partial}{\partial x}(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi)\,dx \quad (2)$$
$$= \frac{i\hbar}{2\mu}(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi)\Big|_{x_1}^{x_2}$$

From (2)

$$j(x,t) = -\frac{i\hbar}{2\mu}(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi) \quad (3)$$

$$\therefore \quad j(x,t) = (-\frac{i\hbar}{2\mu})(2i)\,\mathrm{Im}(\psi^*\frac{\partial\psi}{\partial x}) = (\frac{\hbar}{\mu})\,\mathrm{Im}(\psi^*\frac{\partial\psi}{\partial x})$$

<u>إجابة السؤال 2.b:</u>



The energy equation or Shrodinger equation may be written as:

$$[\frac{d^2}{dx^2} + \frac{2\mu}{\hbar}(E - U(x))]\psi_E = 0 \quad (1)$$

In case $E > U_0$. According the potential regions, equation (1) becomes

$$\left.\begin{array}{lll} \psi_I'' + k_0^2\psi_I = 0, & k_0 = \frac{1}{\hbar}\sqrt{2\mu E}, & x < 0 \\ \psi_{II}'' + k^2\psi_{II} = 0, & k = \frac{1}{\hbar}\sqrt{2\mu(E - U_0)}, & x \geq 0 \end{array}\right\} \quad (2)$$

The general solution of system (2) is

$$\left.\begin{array}{ll} \psi_I(x) = A\exp(ik_0x) + B\exp(ik_0x), & x < 0 \\ \psi_{II}(x) = C\exp(ikx) + D\exp(-ikx), & x \geq 0 \end{array}\right\} \quad (3)$$

Since the beam incident from the left, then D must vanish.

Continuity Conditions

$$\left.\begin{array}{l}\psi_I(0)=\psi_{II}(0)\Rightarrow A+B=C\\ \psi_I''(0)=\psi_{II}''(0)\Rightarrow k_0A-k_0B=kC\end{array}\right\} \qquad (4)$$

From (4)

$B=(\frac{k_0-k}{k_0+k})A$ and $C=(\frac{2k_0}{k_0+k})A$ Thus (3) becomes

$$\psi_E(x)=A\begin{cases}\exp(ik_0x)+(\frac{k_0-k}{k_0+k})\exp(-ik_0x) & x<0\\ (\frac{2k_0}{k_0+k})\exp(ikx) & x\geq 0\end{cases} \qquad (5)$$

Since the reflection coefficient R and the transmission T coefficient satisfy the relations $R=\left|\frac{j_{ref}}{j_{inc}}\right|$ and $T=\left|\frac{j_{tran}}{j_{inc}}\right|$ respectively, where $j(x,t)=(\frac{\hbar}{\mu})\,\mathrm{Im}(\psi^*(x,t)\frac{\partial\psi(x,t)}{\partial x})$.

Now, let us calculate R and T. From the values for A and B with equation (5), one gets

$j_{inc}=\frac{\hbar k_0}{\mu}|A|^2,$ $J_{ref}=-\frac{\hbar k_0}{\mu}|B|^2,$ $J_{tran}=\frac{\hbar k}{\mu}|C|^2$ $\therefore R=\left|\frac{B}{A}\right|^2=(\frac{k_0-k}{k_0+k})^2$

And $T=\frac{k}{k_0}\left|\frac{C}{A}\right|^2=\frac{4k_0k}{(k_0+k)^2}$

Thus

$$.\,R+T=(\frac{k_0-k}{k_0+k})^2+\frac{4k_0k}{(k_0+k)^2}=\frac{k_0^2-2k_0k+k^2+4k_0k}{(k_0+k)^2}=\frac{k_0^2+2k_0k+k^2}{(k_0+k)^2}=1$$

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