جامعة بنها – كلية العلوم – قسم الرياضيات

الفرقة: الرابعة- تربية عام- رياضيات – تخلفات من ثالثة

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مدرس بقسم الرياضيات بكلية العلوم

الامتحان+ نموذج إجابتة

Benha University
Faculty of Science
Dept. of Mathematics

Time : hour and a half
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Integral Equations for Third Year Students

Answer the following questions

Question 1

Obtain solutions for the following integral equations:

(1) $$\varphi(x) = 2x + 4\int_0^x (t-x)\,\varphi(t)\,dt.$$

(2) $$\int_0^x e^{2(x-t)}\,\varphi(t)\,dt = \sin x.$$

Question 2

(a) Investigate the solvability for the following integral equation showing how the Fredholm Alternative is confirmed

$$\varphi(x) = x^3 - x + \lambda\int_{-1}^{1} (x^2 - 2xt)\,\varphi(t)\,dt$$

(b) Evaluate the Resolvent kernel of the kernel

$$k(x,t) = a^{x-t}, \qquad a > 0.$$

Question 3

Use Laplace transformation to solve the following:

$$\varphi_1(x) = e^{2x} + \int_0^x \varphi_2(t)\,dt,$$

$$\varphi_2(x) = 1 - \int_0^x e^{2(x-t)}\,\varphi_1(t)\,dt.$$

Good Luck!

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The answer

Question 1.

(1) $\varphi(x) = 2x + 4\int_0^x (t-x)\varphi(t)\ dt$

$\varphi'(x) = 2 + 4\left[(t-x)\varphi(t)\right]_{t=x} + 4\int_0^x (-1)\varphi(t)\ dt$

$\varphi'(x) = 2 - 4\int_0^x \varphi(t)\ dt \Rightarrow \varphi''(x) = -4\varphi(x)$

$\varphi''(x) + 4\varphi(x) = 0\ ;\ \ \varphi(0) = 0\ ,\ \varphi'(0) = 2$

$\varphi(x) = A\cos 2x + B\sin 2x\ ,\quad \varphi(0) = 0 = A$

$\varphi'(x) = -2A\sin 2x + 2B\cos 2x\ ,\quad \varphi'(0) = 2 = 2B$

solution is there for $\varphi(x) = \sin 2x.$

This can be verified, where we find

$\int_0^x (t-x)\varphi(t)\ dt = \int_0^x (t-x)\sin 2t\ dt =$

$$\left[\frac{x-t}{2}\cos 2t + \frac{1}{4}\sin 2t\right]_0^x = \frac{1}{4}\sin 2t - \frac{x}{2}$$

$$\varphi(x) = 2x + 4\left(\frac{1}{4}\sin 2x - \frac{x}{2}\right) = \sin 2x \quad \text{checks}$$

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(2) $\int_0^x e^{2(x-t)}\,\varphi(t)\,dt = \sin x$

$$\frac{1}{s-2}\phi(s) = \frac{1}{s^2+2} \quad \Rightarrow \quad \phi = \frac{s-2}{s^2+1}$$

Hence $\varphi(x) = \cos x - 2\sin x$

Question 2.

$(a)\quad \varphi(x)=x^3-x+\lambda\int_{-1}^{1}(x^2-2xt)\,\varphi(t)\,dt$

$\phi(x)=\lambda\int_{-1}^{1}(x^2-2xt)\,\phi(t)\,dt=\lambda(Ax^2-2Bx)$

$A=\int_{-1}^{1}\phi(t)\,dt=\lambda\int_{-1}^{1}(At^2-2Bt)\,dt=2\lambda A\int_{0}^{1}t^2dt=\frac{2}{3}\lambda A$

$B=\int_{-1}^{1}t\phi(t)\,dt=\lambda\int_{-1}^{1}(At^3-2Bt^2)\,dt=-4\lambda B\int_{0}^{1}t^2dt=-\frac{4}{3}\lambda B$

Hence $A(1-\frac{2}{3}\lambda)=0$ & $B(1+\frac{4}{3}\lambda)=0\Rightarrow$

eigenvalues $\lambda_1=\frac{3}{2},\ \lambda_2=-\frac{3}{4}$

Now $\varphi(x)=x^3-x+\lambda(ax^2-2bx)$

$a=\int_{-1}^{1}\varphi(t)\,dt=\int_{-1}^{1}(t^3-t)\,dt+\lambda\int_{-1}^{1}(at^2-2bt)dt=\frac{2}{3}\lambda a$

$b=\int_{-1}^{1}t\varphi(t)\,dt=\int_{-1}^{1}(t^4-t^2)\,dt+\lambda\int_{-1}^{1}(at^3-2bt^2)dt=-\frac{4}{15}-\frac{4}{3}\lambda b$

Hence $a(1-\frac{2}{3}\lambda)=0$ & $b(1+\frac{4}{3}\lambda)=-\frac{4}{15}$

these conditions are giving a contradiction only when $\lambda=\lambda_2$.

The solutions to both equation are therefor

$\lambda=\frac{3}{2}$: $\phi=cx^2$ $\varphi=x^3-x+cx^2+\frac{4}{15}x$

$\lambda=-\frac{3}{4}$: $\phi=cx$ no solution φ

$\lambda\neq\lambda_1,\lambda_2$: $\phi=0$ $\varphi=x^3-x+\frac{8\lambda}{5(3+4\lambda)}x$

$$(b) \quad k(x,t) = a^{x-t} = k_1(x,t), \quad a>0$$

$$k_2(x,t) = \int_t^x k(x,t)\,k_1(\sigma,t)\,d\sigma = \int_t^x a^{x-\sigma}a^{6-t}d\sigma = a^{x-t}(x-t)$$

$$k_3(x,t) = \int_t^x a^{x-\sigma}a^{6-t}(\sigma-t)d\sigma = a^{x-t}\frac{(x-t)^2}{2}$$

$$k_4(x,t) = \int_t^x a^{x-\sigma}a^{6-t}\frac{(\sigma-t)^2}{2}d\sigma = a^{x-t}\frac{(x-t)^2}{3!},....$$

$$R(x,t;\lambda) = k_1 + \lambda k_2 + \lambda^2 k_3 + ...$$

$$a^{x-t}\left[1+\lambda(x,t)+\lambda^2\frac{(x-t)^2}{2}+\lambda^3\frac{(x-t)^3}{3!}+....\right]$$

$$= a^{x-t}\ e^{\lambda(x,t)}$$

Question 3.

$$\varphi_1(x) = e^{2x} + \int_0^x \varphi_2(t)\,dt,$$

$$\varphi_2(x) = 1 - \int_0^x e^{2(x-t)}\,\varphi_1(t)\,dt.$$

$$\phi_1 = \frac{1}{s-2} + \frac{1}{s}\phi_2, \qquad \phi_2 = \frac{1}{s} - \frac{1}{s-2}\,\phi_1$$

$$\phi_1 = \frac{3}{s-1} - \frac{2}{s}, \qquad \phi_2 = \frac{3}{s-1} - \frac{2}{s-2}$$

$$\varphi_1(x) = 3e^x - 2, \qquad \varphi_2(x) = 3e^x - 2e^{2x}$$