جامعة بنـها - كلية العلوم – قسم الرياضيـات

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الفصل الدراسي الثانــي

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المادة : ميكانيكا الكم (M424)

أستاذ المادة : د . / خليل محمد خليل محمد

مدرس بقسم الرياضيات بكلية العلوم

صورة من الامتحان+ نموذج إجابته

Faculty of Education
Math. Dept. Benha

Fourth year Math.
Quantum Mechanics (M424)

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Time: 1 hour

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Answer all the following questions:

1.a	Show that: the eigenvalues of a hermitian operator are pure real and its eigenvectors corresponding to unequal eigenvalues are orthogonal.
1.b	State without proof the properties of the projection operator $\hat{P}_i$. Let ψ_i be an orthonormal basis and the operator $\hat{P}_i$ is defined as $\hat{P}_i\phi = \psi_i(\psi_i, \phi)$, find the eigenvalues and the corresponding eigenvectors of $\hat{P}_i$.
1.c	Find the eigenfunctions $\psi(x)$ and the corresponding eigenvalues of the operator: $\hat{A} = i\frac{d}{d\varphi}, \psi(\varphi) = \psi(\varphi + 2\pi)$.
2.a	State the postulates of quantum mechanics.
2.b	Determine the mean value of a mechanical quantity $\hat{L}_z^2$ described by the hermitian operator $\hat{L}_z^2 = -\hbar^2 \frac{\partial^2}{\partial\phi^2}$ in the state $\psi(\phi) = A\sin^2\phi$, $0 < \phi \le 2\pi$.

انظر امتحان الكهربية الديناميكية

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إجابة السؤال 1.a:

Proof:

In the first, we prove that the eigenvalues of a hermitian operator are pure real.

Let $\hat{A}=\hat{A}^{+}$ (i. e. A hermitaian) and $\hat{A}\Psi_a(x)=a\Psi_a(x),\ \Psi_a(x)\neq 0$

then $(\hat{A}\Psi_a,\Psi_a)=(a\Psi_a,\Psi_a)=a^*(\Psi_a,\Psi_a)=a^*\|\Psi_a\|^2$ (i)

but $(\hat{A}\Psi_a,\Psi_a)=(\Psi_a,\hat{A}^{+}\Psi_a)=(\Psi_a,\hat{A}\Psi_a)=(\Psi_a,a\Psi_a)=a(\Psi_a,\Psi_a)$

$=a\|\Psi_a\|^2$ (ii)

from i and ii leads to $a^*\|\Psi_a\|^2 = a\|\Psi_a\|^2 \Rightarrow (a^*-a)\|\Psi_a\|^2=0$

$\because \Psi_a\neq 0 \ \Rightarrow a^*-a=0 \ \Rightarrow a^*=a$ i.e the eigenvalues are pure real

second, suppose

$\hat{A}\Psi_1(x)=a_1\Psi_1(x),\ \Psi_1(x)\neq 0$ *and* $\hat{A}\Psi_2(x)=a_2\Psi_2(x),\ \Psi_2(x)\neq 0;\ a_1\neq a_2$

since $(\Psi_1,\hat{A}\Psi_2)=(\Psi_1,a_2\Psi_2)=a_2(\Psi_1,\Psi_2)$ (iii) also

$(\Psi_1,\hat{A}\Psi_2)=(\hat{A}\Psi_1,\Psi_2)=(a_1\Psi_1,\Psi_2)=a_1(\Psi_1,\Psi_2)$ (iv)

$a_1=a_1^*$ because a_1,a_2 are real then from (iii) and (iv) we have

$a_2(\Psi_1,\Psi_2)=a_1(\Psi_1,\Psi_2)\Rightarrow(a_2-a_1)(\Psi_1,\Psi_2)=0$ but $a_1\neq a_2$

$\Rightarrow(\Psi_1,\Psi_2)=0$ i.e. the eigenvectors are orthogonal.

إجابة السؤال 1.b:

Proof:

The properties of the projection operators $\hat{P}_i$ are:

i- $\hat{P}_i$ is linear ii- $\hat{P}_i$ is hermitian iii- $\hat{P}_i^2=\hat{P}_i$ iv- $\hat{P}_i\hat{P}_j=\hat{P}_j\hat{P}_i=\delta_{ij}\hat{P}_i$

v- $\sum_i \hat{P}_i=\hat{1}$

The eigenvalues of the projection operators : Let $\hat{P}_i\varphi_\lambda=\lambda\varphi_\lambda,\quad \varphi_\lambda\neq 0$

and since $\hat{P}_i$ is linear and $\hat{P}_i^2\varphi_\lambda=\hat{P}_i(\hat{P}\varphi_\lambda)=\hat{P}_i(\lambda\varphi_\lambda)=\lambda(\hat{P}\varphi_\lambda)=\lambda^2\varphi_\lambda$

and also $\hat{P}_i^2=\hat{P}_i \Rightarrow (\lambda^2-\lambda)\varphi_\lambda=\lambda(\lambda-1)\varphi_\lambda=0,\ \varphi_\lambda\neq 0\ \Rightarrow \lambda=0,1.$

** The eigenvectors of the projection operators:

when $\lambda=0$, the corresponding eigenvector is φ_0 then $\hat{P}_i\varphi_0=\psi_i(\psi_i,\varphi_0)=0\varphi_0=0,$

and ψ_i basis then $(\psi_i,\varphi_0)=0\ \Rightarrow \varphi_0=\sum_{j\neq i}\alpha_j\psi_j$ where we can write φ_0 by the basis ψ_i

when $\lambda = 1$, the corresponding eigenvector is φ_1 then $\hat{P}_i\varphi_1 = \psi_i(\psi_i, \varphi_1) = 1\varphi_1 = \varphi_1$,

and ψ_i basis then $(\psi_i, \varphi_1) = k \neq 0 \Rightarrow \varphi_1 = k\psi_i$ (i.e. linearly dependent).

إجابة السؤال 1.c:

To obtain the eigenfunctions and the eigenvalues for the given operator, we suppose

$$\hat{A}\psi = \lambda\psi,\ \psi \neq 0 \Rightarrow i\frac{d}{d\varphi}\psi = \lambda\psi \therefore \frac{d}{d\varphi}\psi(\varphi) = -i\lambda\psi(\varphi) \Rightarrow \int\frac{d\psi}{\psi} = i\lambda\int d\varphi$$

$\Rightarrow \ln(\frac{\psi}{B}) = i\lambda\varphi \Rightarrow \psi = Be^{i\lambda\varphi}$ from the condition $\psi(\varphi) = \psi(\varphi + 2\pi)$

we have $Be^{-i\lambda\varphi} = Be^{-i\lambda(\varphi+2\pi)} = Be^{-i\lambda\varphi}e^{-2i\pi\lambda} \Rightarrow e^{-2i\pi\lambda} = 1 \quad \therefore \cos(-2\pi\lambda) + i\sin(-2\pi\lambda) = 1$

$\Rightarrow \cos(2\pi\lambda) - i\sin(2\pi\lambda) = 1$ by comparison $2\pi\lambda = 2n\pi,\quad n = 0,1,2,.....\quad \lambda = n$ and the eigenfns.

$\psi(\varphi) = Be^{-in\varphi},\quad n = 0,1,2,.....$

إجابة السؤال 2.a:

*The postulates of quantum mechanics are:

1)-Postulate I: Every physical state of a dynamical system (a particle) is represented at a given instant of time t by normed vector $|\psi\rangle_t$ in H. It is assumed that the state vector contains all the information which one can know about the state of the system at that instant of time. ψ and $e^{i\delta}\psi$ where $\delta^* = \delta$ represent the same physical state.

2)- Postulate II: To every dynamical variable A there corresponds an observable $\hat{A}$. The observable $\hat{x}$ and $\hat{p}$ must satisfy $[\hat{x}, \hat{p}] = i\hbar$. The rules for constructing the observable $\hat{A}$ corresponding to the dynamical variable A, in the $x - rep$ are as follows:

$(i) x \rightarrow \hat{x} = x,\ t \rightarrow \hat{t} = t,\ p \rightarrow \hat{p} = -i\hbar\frac{d}{dx}$

$(ii) A(x, p, t) \rightarrow \hat{A} = A(x, -i\hbar\frac{d}{dx}, t).$

3)- Postulate III: If a particle is in state $|\psi\rangle_t$, a measurement of a dynamical variable A which is represented by the observable $\hat{A}$

$\hat{A}|\varphi_n\rangle = a_n|\varphi_n\rangle,\ \langle\varphi_n|\varphi_n\rangle = \delta_{nm},\quad \hat{1}_a = \sum_i |\varphi_i\rangle\langle\varphi_i|$ will

*yield one of the eigenvalues a_i with probability

$$\rho_\psi(a_i) = \frac{|\langle\varphi_i|\psi\rangle|^2}{\langle\psi|\psi\rangle}$$

** If the result of measurement is a_k, then the state of the system will change from $|\psi\rangle$ to $|\varphi_k\rangle$ as a result of measurement.

4)- Postulate IV: The state function $\psi(x,t)$ describing the state of a dynamical system whose Hamiltonian is $\hat{H}$ obeys the following" Schrodinger time-dependent" equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t)=\hat{H}\psi(x,t)$$

إجابة السؤال 2.b:

$\because \psi(\varphi)=A\sin^2(\varphi)$ describe the system i.e. normed

$$\therefore \|\psi(\varphi)\|^2=1\Rightarrow\int_0^{2\pi}|A|^2\sin^4(\varphi)d\varphi=1\Rightarrow|A|^2\int_0^{2\pi}\sin^4(\varphi)d\varphi=1$$

$$\because \sin^2(\varphi)=\frac{1}{2}(1-\cos(2\varphi)),\ \cos^2(\varphi)=\frac{1}{2}(1+\cos(2\varphi))\ \therefore\frac{|A|^2}{4}\int_0^{2\pi}(1-\cos(2\varphi))(1+\cos(2\varphi))d\varphi$$

$$=\frac{|A|^2}{4}\int_0^{2\pi}(1-2\cos(2\varphi)+\cos^2(2\varphi))d\varphi=\frac{|A|^2}{4}\int_0^{2\pi}(1-2\cos(2\varphi)+\frac{1}{2}(1+\cos(4\varphi)))d\varphi$$

$$=\frac{|A|^2}{4}\int_0^{2\pi}(\frac{3}{2}-2\cos(2\varphi)+\frac{1}{2}\cos(4\varphi))d\varphi=\frac{|A|^2}{4}[\frac{3}{2}\varphi-\sin(2\varphi)+\frac{1}{8}\sin(4\varphi)]_0^{2\pi}$$

$$\therefore\frac{|A|^2}{4}[\frac{3}{2}(2\pi)-0+\frac{1}{8}(0)-0+0-0]=\frac{3\pi|A|^2}{4}=0\ \therefore|A|^2=\frac{4}{3\pi}\Rightarrow A=\frac{2}{\sqrt{3\pi}}$$

$$\therefore\psi(\varphi)=\frac{2}{\sqrt{3\pi}}\sin^2(\varphi),\qquad 0<\varphi<2\pi$$

$$\therefore\langle L_Z^2\rangle_\psi=\langle\psi|L_Z^2|\psi\rangle=\int_0^{2\pi}\frac{2}{\sqrt{3\pi}}\sin^2(\varphi).(-\hbar^2\frac{d^2}{d\varphi^2}\frac{2}{\sqrt{3\pi}}\sin^2(\varphi))d\varphi$$

$$=\frac{4}{3\pi}(-\hbar^2)\int_0^{2\pi}\sin^2(\varphi).\frac{d^2}{d\varphi^2}\sin^2(\varphi)d\varphi=\frac{-4\hbar^2}{3\pi}\int_0^{2\pi}\sin^2(\varphi).[2(\cos^2(\varphi)-\sin^2(\varphi)]d\varphi$$

$$=\frac{-4\hbar^2}{3\pi}\int_0^{2\pi}\sin^2(\varphi).2\cos(2\varphi)d\varphi=\frac{-4\hbar^2}{3\pi}\int_0^{2\pi}\frac{1}{2}(1-\cos(2\varphi)).2\cos(2\varphi)d\varphi$$

$$=\frac{-4\hbar^2}{3\pi}\int_0^{2\pi}(\cos(2\varphi)-\cos^2(2\varphi)d\varphi=\frac{-4\hbar^2}{3\pi}\int_0^{2\pi}(\cos(2\varphi)-\frac{1}{2}(1+\cos(4\varphi)))d\varphi$$

$$=\frac{-4\hbar^2}{3\pi}[\frac{1}{2}\sin(2\varphi)-\frac{1}{2}\varphi-\frac{1}{8}\sin(4\varphi))]_0^{2\pi}=\frac{-4\hbar^2}{3\pi}[0-\frac{1}{2}2\pi-0)]=\frac{4\hbar^2}{3}.$$

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