==

The perfect Answer:

(1) Solution. (T1) Obviously $\phi \in \rho$ and since $p \in X$, so $X \in \rho$.

(T2) Let $G, H \in \rho$. Then $p \in G$ and $p \in H$,

therefore, $p \in G \cap H$.

Hence, $G \cap H \in \rho$.

(T3) Let $H_i \in \rho$, where $i \in I$. Then $p \in H_i$, $\forall$ $i \in I$,

therefore, $p \in \bigcup_{i \in I} H_i$.

Hence, $\bigcup_{i \in I} H_i \in \rho$.

That is ρ is a topology on X.

(2) (i) Let (X, τ) be a topological space and $A, B \subseteq X$ .A point $\alpha \in A$ is called an interior point of A if there is an open set G such that $\alpha \in G \subseteq A$. The set of all interior points of a set A is called the interior of A and denoted by int A.

From definition we have

int $A \cap$ ext $A = \phi$, int $A \cap b(A) = \phi$ and ext $A \cap b(A) = \phi$.

Also,

$$\text{int } A \cup \text{ext } A \cup b(A) = \text{int } A \cup \text{ext } A \cup [\,(\text{int } A)^c \cap (\text{ext } A)^c]$$
$$= [\text{int } A \cup \text{ext } A \cup (\text{int } A)^c] \cap [\text{int } A \cup \text{ext } A \cup (\text{ext } A)^c]$$
$$= [\,X \cup \text{ext } A] \cap [\,\text{int } A \cup X]$$
$$= X \cap X$$
$$= X.$$

ii) It follows from part (i) that $A' \subseteq (A \cup B)'$ and $B' \subseteq (A \cup B)'$. Then

$$A' \cup B' \subseteq (A \cup B)', \qquad (1)$$

Conversely, we prove that $(A \cup B)' \subseteq A' \cup B'$.

Take $\alpha \notin A' \cup B'$, we have $\alpha \notin A'$ and $\alpha \notin B'$

and hence there exist neighborhoods $V, W \in \mathcal{N}_\alpha$

such that $(V - \{\alpha\}) \cap A = \phi$ and $(W - \{\alpha\}) \cap B = \phi$.

But $V \cap W \in \mathcal{N}_\alpha$ satisfies

$[(V \cap W) - \{\alpha\}] \cap (A \cup B) = \phi$. Then

$\alpha \notin (A \cup B)'$, which proves that

$(A \cup B)' \subseteq A' \cup B'$, (2)

From (1), (2) we get

$(A \cup B)' = A' \cup B'$.

(3) The closed sets of (X, τ) are ϕ, X, $\{\beta, \theta\}$ and $h(\phi) = \phi$, $h(X) = Y$ and $h(\{\beta, \theta\}) = \{a, b\} = Y$ are closed of (Y, σ), this shows that h is closed.

Also, the set $\{\alpha\}$ is an open of (X, τ), but $h(\{\alpha\}) = \{b\}$ is not open of (Y, σ), this shows that h is not open function.

Finally, since the set $\{a\} \in \sigma$, but $h^{-1}(\{a\}) = \{\theta\} \notin \tau$, then this shows that h is not continuous function.

(4) The function $d : \Re \times \Re \longrightarrow \Re^*$ defined by $d(x, y) = |x - y|$, where $x, y \in \Re$, is a metric on $\Re$, because for all $x, y, z \in \Re$, we have

(1) $d(x, y) = |x - y| > 0$ and $d(x, x) = |x - x| = 0$

(2) $d(x, y) = |x - y| = |y - x| = d(y, x)$

(3) $d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z|$

$= d(x, y) + d(y, z)$.

The open sphere $S(0, 6)$ with center $p = 0 \in \Re$ and radius $\delta = 6$, is

$$
\begin{aligned}
S(0, 6) &= \{x \in X : d(x, 0) < 6\} \\
&= \{x \in X : |x - 0| < 6\} \\
&= \{x \in X : |x| < 6\} \\
&= \{x \in X : -6 < x < 6\} \\
&=]-6, 6[.
\end{aligned}
$$